Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 329 (2007) 904–912
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Multiplicity of time-periodic solutions and their stabilities for quasigeostrophic equations

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Received 13 February 2006
Available online 8 August 2006
Submitted by W. Layton

Abstract

In this article, we study the quasigeostrophic equation, which is a prototypical geophysical fluid model. We will show the existence of multiple time-periodic solutions and study their stabilities. We also prove the exponential decaying of solutions to the initial value problem with small initial data.

Keywords: Quasigeostrophic fluid model; Time-periodic motion; Positive solution; Monotonicity; Stability; Exponential decay

1. Introduction

The two-dimensional barotropic quasigeostrophic equation considered in this article is given by

$$\Delta\psi_t + J(\psi, \Delta\psi) + \beta\psi_x = \nu\Delta^2\psi - r\Delta\psi + f(x, y, t), \tag{1.1}$$

where $\psi(x, y, t)$ is the stream function, $\beta > 0$ is the meridional gradient of the Coriolis parameter, $\nu > 0$ is the viscous dissipation constant, $r > 0$ is the Ekman dissipation constant, and $f(x, y, t)$ is the wind forcing. Moreover, $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplacian operator in the xy-plane, $x, y \in D$ an arbitrary bounded planar domain with piecewise smooth boundary, and $J(f, g) = f_x g_y - f_y g_x$ is the Jacobi operator.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.07.043

This equation is derived as an approximation of the rotating shallow water equations by the conventional asymptotic expansion in small Rossby number [1,2]. Schochet [6] has recently shown that quasigeostrophy is a valid approximation of the rotating shallow water equations in the limit of zero Rossby number, i.e., at asymptotically high rotation rate.

Let $\omega = \Delta\psi$, then the quasigeostrophic equation becomes the following system of equations,

$$\begin{aligned} &\omega_t + J(\psi,\omega) + \beta\psi_x = \nu\Delta\omega - r\omega + f(x,y,t),\\ &\Delta\psi = \omega, \end{aligned} \tag{1.2}$$

with the following boundary and initial conditions, which satisfies appropriate compatible conditions:

$$\begin{aligned} &\psi(x,y,t) = g_1(x,y,t) \quad \text{on } \partial D,\\ &\omega(x,y,t) = g_2(x,y,t) \quad \text{on } \partial D,\\ &\omega(x,y,0) = \omega_0(x,y). \end{aligned} \tag{1.3}$$

In [3] the author has shown the existence of time-periodic solution for the case $g_1(x,y,t) = g_2(x,y,t) = 0$ when the Coriolis parameter and the Ekman dissipation constant satisfy the technical condition $r + \frac{\pi\nu}{|D|} > \frac{1}{2}\beta(\frac{|D|}{\pi} + 1)$, here D is the area of domain D.

In this article we will study the existence of periodic solutions for a more general quasi-geostrophic equation with both boundary and forcing are periodic and depend on ω,

$$\begin{aligned} &\omega_t + J(\psi,\omega) + \beta\psi_x = \nu\Delta\omega - r\omega + f(x,y,t,\omega),\\ &\Delta\psi = \omega, \end{aligned} \tag{1.4}$$

with the following boundary and initial conditions, which satisfies appropriate compatible conditions:

$$\begin{aligned} &\psi(x,y,t) = g_1(x,y,t,\omega) \quad \text{on } \partial D,\\ &\omega(x,y,t) = g_2(x,y,t,\omega) \quad \text{on } \partial D,\\ &\omega(x,y,0) = \omega_0(x,y). \end{aligned} \tag{1.5}$$

We will show that, when $f(x,y,t,\omega)$, $g_1(x,y,t,\omega)$, $g_2(x,y,t,\omega)$ are positive and periodic in time, the quasigeostrophic equations have at least one time-periodic solution $(\omega(x,y,t), \psi(x,y,t))$ with $\omega(x,y,t) \geqslant 0$. Our results do not require the technical condition imposed on r and β.

We will also show that the time-periodic solution is exponentially stable in the sense that, if $\omega_1(x,y,t)$ is any solution with $\omega_1(x,y,0)$ in a neighborhood of $\omega(x,y,0)$, then $|\omega_1(x,y,t) - \omega(x,y,t)| \leqslant Ce^{-at}$, as $t \to \infty$. A corollary of this result is that, when $f(x,y,t,0) = g_(x,y,t,0) = g_2(x,y,t,0) = 0$, any solution to the initial value problem with small initial data will decay exponentially as $t \to \infty$.

Furthermore, we also show that under right condition on $f(x,y,t,z)$ the equation admits at least two stable time-periodic solutions.

2. Main results

In this section we will state and prove our main results.

First the positive solution result, this result indicates that the quasigeostrophic equation enjoys a maximum principle like a heat equation [5].

Theorem 2.1. *Suppose $f(x,y,t)$, $g_1(x,y,t)$, $g_2(x,y,t)$, and ω_0 is nonnegative, then the solution $\omega(x,y,t)$ of* (1.2)–(1.3) *is nonnegative for all time $t>0$.*

Proof. The existence is standard, see [7–9]. We just prove the positiveness. To this end, let $u(x,y,t)=e^{-\lambda t}\omega(x,y,t)$, and $\Psi(x,y,t)=e^{-\lambda t}\psi(x,y,t)$, we have

$$u_t+\lambda u+e^{\lambda t}J(\Psi,u)+\beta\Psi=\nu\Delta u-ru+e^{-\lambda t}f(x,y,t)$$

and

$$\Delta\Psi=u.$$

Now if $\omega(x_0,y_0,t_0)<0$, at some interior point $(x_0,y_0,t_0)\in(0,T)\times D$. We have u reaches its negative minimum at some interior $(x_1,y_1,t_1)\in(0,T)\times D$. So

$$u_t(x_1,y_1,t_1)=u_x(x_1,y_1,t_1)=u_y(x_1,y_1,t_1)=0,$$

this indicates that, at (x_1,y_1,t_1),

$$J(\Psi,u)=0.$$

Furthermore, we have

$$\Delta u(x_1,y_1,t_1)\geqslant 0.$$

Using the equation for u we have

$$\begin{aligned}\lambda u(x_1,y_1,t_1)+\beta\Psi_x(x_1,y_1,t_1)&=\nu\Delta u(x_1,y_1,t_1)-ru(x_1,y_1,t_1)+e^{-\lambda t}f(x_1,y_1,t_1)\\&\geqslant -ru(x_1,y_1,t_1)+e^{-\lambda t}f(x_1,y_1,t_1).\end{aligned}$$

On the other hand, from standard theory on elliptic partial differential equation [4] of second-order we have

$$\bigl|\psi_x(x_1,y_1,t_1)\bigr|\leqslant C\bigl(|\omega|(t_1)+|g_2|(t_1)\bigr),$$

where $|h|(t)$ is the maximum value of a function $g(x,y,t)$ over D for each fixed t.

Since $\omega(x_1,y_1,t_1)<0$ we can choose λ large enough so that

$$0>\lambda\omega(x_1,y_1,t_1)+\beta C\bigl(|\omega|(t_1)+|g_2|(t_1)\bigr)+r\omega(x_1,y_1,t_1),$$

which contradicts with

$$\lambda\omega(x_1,y_1,t_1)+\beta C\bigl(|\omega|(t_1)+|g_2|(t_1)\bigr)+r\omega(x_1,y_1,t_1)\geqslant f(x_1,y_1,t_1)\geqslant 0.$$

Therefore u, and hence ω have to be nonnegative. □

A corollary of this theorem is the following comparison result.

Corollary 2.2. *Suppose that $f^{(1)}(x,y,t)\geqslant f^{(2)}(x,y,t)$, $g_1^{(1)}(x,y,t)\geqslant g_1^{(2)}(x,y,t)$, $g_2^{(1)}(x,y,t)\geqslant g_2^{(2)}(x,y,t)$, and $\omega_0^{(1)}(x,y)\geqslant\omega_0^{(2)}(x,y)$, for all x,y,t.*

Let $\omega^{(i)}(x,y,t)$, $\psi^{(i)}(x,y,t)$, $i=1,2$, be solutions of

$$\begin{aligned}&\omega_t^{(i)}+J\bigl(\psi^{(i)},\omega^{(i)}\bigr)+\beta\psi_x^{(i)}=\nu\Delta\omega^{(i)}-r\omega^{(i)}+f^{(i)}(x,y,t),\\&\Delta\psi^{(i)}=\omega^{(i)},\end{aligned}\tag{2.1}$$

with boundary and initial conditions

$$\psi^{(i)}(x,y,t)=g_1^{(i)}(x,y,t) \quad \textit{on } \partial D,$$
$$\omega^{(i)}(x,y,t)=g_2^{(i)}(x,y,t) \quad \textit{on } \partial D,$$
$$\omega^{(i)}(x,y,0)=\omega_0^{(i)}(x,y).$$

We know that $\omega^{(1)}(x,y,t)\geqslant\omega^{(2)}(x,y,t)$ *for all* x,y,t.

Proof. Let

$$\delta\omega(x,y,t)=\omega^{(1)}(x,y,t)-\omega^{(2)}(x,y,t), \qquad \delta\psi(x,y,t)=\psi^{(1)}(x,y,t)-\psi^{(2)}(x,y,t),$$
$$\delta f(x,y,t)=f^{(1)}(x,y,t)-f^{(2)}(x,y,t), \qquad \delta g_1(x,y,t)=g_1^{(1)}(x,y,t)-g_1^{(2)}(x,y,t),$$
$$\delta g_2(x,y,t)=g_2^{(1)}(x,y,t)-g_2^{(2)}(x,y,t), \qquad \delta\omega_0(x,y)=\omega_0^{(1)}(x,y)-\omega_0^{(2)}(x,y),$$

we have

$$\begin{aligned}&\delta\omega_t+J\big(\psi^{(1)},\delta\omega\big)+J\big(\delta\psi,\omega^{(2)}\big)+\beta\delta\psi_x=\nu\Delta\delta\omega-r\delta\omega+\delta f(x,y,t),\\&\Delta\delta\psi=\delta\omega,\end{aligned} \tag{2.2}$$

with boundary and initial conditions

$$\delta\psi(x,y,t)=\delta g_1(x,y,t) \quad \text{on } \partial D,$$
$$\delta\omega(x,y,t)=\delta g_2(x,y,t) \quad \text{on } \partial D,$$
$$\delta\omega(x,y,0)=\delta\omega_0(x,y).$$

Proceeding as in the proof of Theorem 2.1, and noticing that for all $t\geqslant 0$,

$$\big|J\big(\delta\psi,\omega^{(2)}\big)+\beta\delta\psi_x\big|(t)\leqslant C|\delta\psi|(t),$$

we can prove our result. □

Let $\overline{D}=D\cup\partial D$ and $\mathcal{X}=C^{1,2}([0,\infty)\times\overline{D})\cap C([0,\infty)\times\overline{D})$. Hereafter, we require that $f(x,y,\cdot)\in C^\alpha(\overline{D})$, $g_1(x,y,\cdot), g_2(x,y,\cdot)\in C^\alpha(\partial D)$ are all time periodic functions, and ∂D is of class $C^{1+\alpha}$.

We define the upper and lower solutions for the quasigeostrophic equations below.

Definition 2.3. A function $\bar{\boldsymbol{u}}=(\overline{\omega},\overline{\psi})$ is called an upper solution of (1.4)–(1.5) in $\mathcal{X}$ if

$$\begin{aligned}&\overline{\omega}_t+J(\overline{\psi},\overline{\omega})+\beta\overline{\psi}_x-\nu\Delta\overline{\omega}+r\overline{\omega}\geqslant f(x,y,t,\overline{\omega}),\\&\Delta\overline{\psi}=\overline{\omega},\\&\overline{\psi}(x,y,t)\geqslant g_1(x,y,t,\overline{\psi}) \quad \text{on } \partial D,\\&\overline{\omega}(x,y,t)\geqslant g_2(x,y,t,\overline{\omega}) \quad \text{on } \partial D,\\&\overline{\omega}(x,y,0)\geqslant\overline{\omega}(x,y,T).\end{aligned} \tag{2.3}$$

Similarly, $\underline{\boldsymbol{u}}=(\underline{\omega},\underline{\psi})$ is called a lower solution if it satisfies the inequalities in (2.3) in reversed order.

Lemma 2.4. *Suppose that*

(1) $f(x,y,t,z)$, $g_1(x,y,t,z)$*, and* $g_2(x,y,t,z)$ *are positive and nondecreasing in z, and*

(2) *there exists* $M > 0$ *such that,* $\forall (x, y, t, z) \in [0, T] \times \overline{D} \times [0, M]$,

$$M > \max\left\{\frac{f(x, y, t, z)}{r}, g_1\left(x, y, t, z\left(\frac{1}{2}y^2 + 1\right)\right), g_2(x, y, t, z)\right\},$$

we have $\underline{\boldsymbol{u}}_0 = (0, 0)$ *is a lower solution of* (1.4)–(1.5) *and* $\underline{\boldsymbol{u}}_M = (M, M(\frac{1}{2}y^2 + 1))$ *is an upper solution of* (1.4)–(1.5).

Proof. It is easy to see that $\underline{\boldsymbol{u}}_0 = (0, 0)$ is a lower solution of (1.4)–(1.5). To show that $\underline{\boldsymbol{u}}_M = (M, M(\frac{1}{2}y^2 + 1))$ is an upper solution of (1.4)–(1.5) one just needs to notice that, for $\overline{\psi} = M(\frac{1}{2}y^2 + 1)$ we have $\Delta\overline{\psi} = M$ and $\overline{\psi}_x = 0$. $\square$

A pair of upper and lower solutions $\bar{\boldsymbol{u}}, \underline{\boldsymbol{u}}$ is said to be ordered if $\bar{\boldsymbol{u}} \geqslant \underline{\boldsymbol{u}}$ on $\overline{D}$. Let $\boldsymbol{J} \equiv \{\boldsymbol{u} \in C([0, \infty) \times \overline{D}); 0 \leqslant \boldsymbol{u} \leqslant M\}$, where M is the constant specified in Lemma 2.4.

Theorem 2.5. *Under assumption of Lemma* 2.4*, the quasigeostrophic equations* (1.4)–(1.5) *have both maximal periodic solution* $\bar{\boldsymbol{u}}$ *and a minimal periodic solution* $\underline{\boldsymbol{u}}$ *that satisfy*

$$0 \leqslant \underline{\boldsymbol{u}} \leqslant \bar{\boldsymbol{u}}.$$

Here we say that $\bar{\boldsymbol{u}}$ *is maximal and* $\underline{\boldsymbol{u}}$ *is minimal in the sense that, for any periodic solution* $\boldsymbol{u} \in \boldsymbol{J}$*, we have*

$$\underline{\boldsymbol{u}} \leqslant \boldsymbol{u} \leqslant \bar{\boldsymbol{u}}.$$

Proof. Starting from either $u^{(0)} = \underline{\boldsymbol{u}}_0$ or $u^{(0)} = \bar{\boldsymbol{u}}_M$ as an initial iteration we construct a sequence $\{\boldsymbol{u}^{(m)} = (\omega^{(m)}, \psi^{(m)})\}$, $m = 1, 2, 3, \ldots,$ by solving

$$\begin{aligned}
&\omega_t^{(m)} + J\big(\psi^{(m)}, \omega^{(m)}\big) + \beta\psi_x^{(m)} = \nu\Delta\omega^{(m)} - r\omega^{(m)} + f\big(x, y, t, \omega^{(m-1)}\big),\\
&\Delta\psi^{(m)} = \omega^{(m)},\\
&\psi^{(m)}(x, y, t) = g_1\big(x, y, t, \psi^{(m-1)}\big) \quad \text{on } \partial D,\\
&\omega^{(m)}(x, y, t) = g_2\big(x, y, t, \omega^{(m-1)}\big) \quad \text{on } \partial D,\\
&\omega^{(m)}(x, y, 0) = \omega^{(m-1)}(x, y, T).
\end{aligned}$$

By the assumptions on f, g_1, g_2, suppose that $0 \leqslant |\omega^{(m)}(x, y, t)| \leqslant M_1$ for all m, we have

$$\Delta\psi^{(m)} = \omega^{(m)},$$

multiplying both sides by $\Delta\psi^{(m)}$ and integrating over $\overline{D}$ we have

$$\int\limits_{\overline{D}} \big|\Delta\psi^{(m)}\big|^2\, dx\, dy = \int\limits_{\overline{D}} \omega^{(m)}\Delta\psi^{(m)}\, dx\, dy \leqslant \epsilon \int\limits_{\overline{D}} \big|\Delta\psi^{(m)}\big|^2\, dx\, dy + \frac{2M_1}{\epsilon}.$$

So $\{\psi^{(m)}(t)\} \subset H^2(\overline{D})$ is a bounded sequence. By Sobolev embedding we have $\{\psi^{(m)}(t)\} \subset C^{1,\alpha}(\overline{D})$ is a bounded sequence.

Now a bootstrap type argument and the results on the nonlinear parabolic equations can ensure us that $\{\psi^{(m)}\}$, $\{\omega^{(m)}\}$ are bounded sequences in X, provided that $0 \leqslant |\omega^{(m)}(x, y, t)| \leqslant M_1$ for all m.

Denote the sequence by $\{\bar{\boldsymbol{u}}^{(m)}\}$ when $u^{(0)} = \underline{\boldsymbol{u}}_0$ and $\{\bar{\boldsymbol{u}}^{(m)}\}$ when $u^{(0)} = \bar{\boldsymbol{u}}_M$.

By Corollary 2.2 we know that

$$\underline{\boldsymbol{u}}_0 \leqslant \underline{\boldsymbol{u}}^{(m)} \leqslant \underline{\boldsymbol{u}}^{(m+1)} \leqslant \bar{\boldsymbol{u}}^{(m+1)} \leqslant \bar{\boldsymbol{u}}^{(m)} \leqslant \bar{\boldsymbol{u}}_M.$$

Hence $0 \leqslant \underline{\omega}^{(m)} \leqslant \overline{\omega}^{(m)}(x, y, t) \leqslant M_1$ for all m. The above discussion shows that $\{\bar{\boldsymbol{u}}^{(m)}\}$ and $\{\underline{\boldsymbol{u}}^{(m)}\}$ are bounded sequences in X, so we have

$$\bar{\boldsymbol{u}} = \lim_{m\to\infty} \bar{\boldsymbol{u}}^{(m)} \in X \quad \text{and} \quad \underline{\boldsymbol{u}} = \lim_{m\to\infty} \underline{\boldsymbol{u}}^{(m)} \in X$$

are two solutions to (2.4).

To show that both $\bar{\boldsymbol{u}}$ and $\underline{\boldsymbol{u}}$ are periodic solutions, we need to show that $\omega(x, y, t) = \omega(x, y, t + T)$ for all $t > 0$, here we use $\omega(x, y, t)$ to denote either $\overline{\omega}(x, y, t)$ or $\underline{\omega}(x, y, t)$.

Let $\delta\omega(x, y, t) = \omega(x, y, t) - \omega(x, y, t + T)$ we have

$$\begin{aligned} &\delta\omega_t + J(\psi, \delta\omega) + J(\delta\psi, \omega) + \beta\delta\psi_x = \nu\Delta\delta\omega - r\delta\omega + \delta f(x, y, t)\delta\omega, \\ &\Delta\delta\psi = \delta\omega, \end{aligned} \tag{2.4}$$

with boundary and initial conditions

$$\begin{aligned} &\delta\psi(x, y, t) = \delta g_1(x, y, t)\delta\psi(x, y, t) \quad \text{on } \partial D, \\ &\delta\omega(x, y, t) = \delta g_2(x, y, t)\delta\omega \quad \text{on } \partial D, \\ &\delta\omega(x, y, 0) = 0, \end{aligned}$$

where $\delta f(x, y, t) = \frac{\partial f(x,y,t,\eta))}{\partial z} \geqslant 0$, with $\eta(x, y, t)$ being the intermediate values between $\omega(x, y, t)$ and $\omega(x, y, t + T)$. Similarly, we have $\delta g_1(x, y, t) \geqslant 0$, $\delta g_2(x, y, t) \geqslant 0$. Now proceeding as in Theorem 2.1, we have $\delta\omega \geqslant 0$, replacing $\delta\omega$ by $-\delta\omega$ we can get $\delta\omega \leqslant 0$, we have $\delta\omega(x, y, t) \equiv 0$. So $\omega(x, y, t)$ is a periodic function in t with period T, hence $\psi(x, y, t)$ is a periodic function in t with period T.

To prove that $\bar{\boldsymbol{u}}$ is maximal and $\underline{\boldsymbol{u}}$ is minimal in the sense that, for any periodic solution $\boldsymbol{u} \in \boldsymbol{J}$, we have

$$\underline{\boldsymbol{u}} \leqslant \boldsymbol{u} \leqslant \bar{\boldsymbol{u}}.$$

To this end, one just needs to see that we can regard $\boldsymbol{u}$ as both upper and lower solutions at the same time and we will have $\boldsymbol{u} \geqslant \underline{\boldsymbol{u}}^{(m)} \geqslant \underline{\boldsymbol{u}}_0$, we have $\boldsymbol{u} \geqslant \underline{\boldsymbol{u}}$. Similarly, we have $\boldsymbol{u} \leqslant \bar{\boldsymbol{u}}$. □

Finally, we will prove the following stability result for the periodic solutions.

Theorem 2.6. *Suppose that $f(x, y, t, z) \geqslant 0$ is monotonic in z, and there exist $b < r$ and $\delta_0 > 0$ such that*

$$\left|\frac{\partial f(x, y, t\omega^* + \delta)}{\partial z} + y\omega_x^*\right| \leqslant b$$

for all $\delta \leqslant \delta_0$, where (ω^*, ψ^*) is the periodic solution to the equation

$$\begin{aligned} &\omega_t + J(\psi, \omega) + \beta\psi_x = \nu\Delta\omega - r\omega + f(x, y, t, \omega), \\ &\Delta\psi = \omega, \end{aligned} \tag{2.5}$$

with boundary and initial conditions

$$\psi(x,y,t)=g_1(x,y,t) \quad \text{on } \partial D,$$
$$\omega(x,y,t)=g_2(x,y,t) \quad \text{on } \partial D,$$
$$\omega(x,y,0)=\omega_0(x,y).$$

Then, for any $\omega_0(x,y)$,

$$\omega^*(x,y,0)-\delta_0 b \leqslant \omega_0(x,y) \leqslant \omega^*(x,y,0)+\delta_0 b,$$

if $(\omega(x,y,t),\psi(x,y,t))$ is a solution to (2.5) *with $\omega(x,y,0)=\omega_0(x,y)$, we have*

$$\left|\omega(x,y,t)-\omega^*(x,y,t)\right| \leqslant \delta_0 e^{(b-r)t}$$

and

$$\left|\psi(x,y,t)-\psi^*(x,y,t)\right| \leqslant C\delta_0 e^{(b-r)t},$$

where C depends on Ω and $g_1(x,y,t)$.

Proof. Let $\underline{\boldsymbol{u}}=(\omega^*-\delta_0 e^{(b-r)t}, \psi^*-\frac{1}{2}y^2\delta_0 e^{(b-r)t})$ and $\bar{\boldsymbol{u}}=(\omega^*+\delta_0 e^{(b-r)t}, \psi^*+\frac{1}{2}y^2\delta_0 e^{(b-r)t})$.

We can show that $\underline{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}$ are upper and lower solutions of Eq. (2.5), provided that $|y\omega_x^*| \leqslant r-b$. To show that $\underline{\boldsymbol{u}}$ is a lower solution, let $\underline{\omega}=\omega^*-\delta_0 e^{(b-r)t}$, $\underline{\psi}=\psi^*-\frac{1}{2}y^2\delta_0 e^{(b-r)t}$.

It is easy to see that

$$\Delta\underline{\psi}=\Delta\left(\psi^*-\frac{1}{2}y^2\delta_0 e^{(b-r)t}\right)=\Delta\psi^*-\delta_0 e^{(b-r)t}=\underline{\omega},$$
$$\underline{\psi}(x,y,t)=g_1(x,y,t)-\delta_0 e^{(b-r)t} \leqslant g_1(x,y,t) \quad \text{on } \partial D,$$
$$\underline{\omega}(x,y,t)=g_2(x,y,t)-\delta_0 e^{(b-r)t} \leqslant g_2(x,y,t) \quad \text{on } \partial D,$$
$$\underline{\omega}(x,y,0)=\omega^*(x,y,0)-\delta_0=\omega^*(x,y,T)-\delta_0 \leqslant \omega^*(x,y,T)-\delta_0 e^{(b-r)T}=\underline{\omega}(x,y,T).$$

Now, for $\underline{\omega}$, we have

$$\begin{aligned}
&\underline{\omega}_t+J(\underline{\psi},\underline{\omega})+\beta\underline{\psi}_x-\nu\Delta\underline{\omega}+r\underline{\omega} \\
&\quad=\omega_t^*+J(\psi^*,\omega^*)+\beta\psi_x^*-\nu\Delta\omega^*-r\omega^*+y\delta_0 e^{(b-r)t}\omega_x^* \\
&\qquad+r\delta_0 e^{(b-r)t}-(b-r)\delta_0 e^{(b-r)t} \\
&\quad=f(x,y,t,\omega^*)+y\delta_0 e^{(b-r)t}\omega_x^*-r\delta_0 e^{(b-r)t}-(b-r)\delta_0 e^{(b-r)t} \\
&\quad=\left(\frac{\partial f(x,y,t,\xi)}{\partial z}+y\omega_x^*-b\right)\delta_0 e^{(b-r)t}+f(x,y,t,\underline{\omega}) \\
&\quad\leqslant f(x,y,t,\underline{\omega}).
\end{aligned}$$

This shows that $\underline{\boldsymbol{u}}$ is a lower solution. Similarly, we can show that $\bar{\boldsymbol{u}}$ is an upper solution.

Using the comparison result of Corollary 2.2, we have that

$$\omega^*-\delta_0 e^{(b-r)t} \leqslant \omega \leqslant \omega^*+\delta_0 e^{(b-r)t}.$$

Elementary result on the elliptic equations show that

$$\left|\psi(x,y,t)-\psi^*(x,y,t)\right| \leqslant C\delta_0 e^{(b-r)t},$$

with C depending on Ω and $g_1(x,y,t)$. □

Remark 2.7. Notice, if $f(x,y,t,0)=0$ and $g_1(x,y,t)=g_2(x,y,t)=0$ we have the trivial period solution, $\omega(x,y,t)=\psi(x,y,t)=0$. So if $|\frac{\partial f(x,y,t\delta)}{\partial z}|\leqslant b$ the condition in Theorem 2.6 is satisfied and we thus show that when the initial data is small, the solution will decay exponentially.

Corollary 2.8. *Suppose that $f(x,y,t,z)\geqslant 0$ is monotonic in z, and there exist $b<r$ and $\delta_0>0$ such that*

$$\left|\frac{\partial f(x,y,t\delta)}{\partial z}\right|\leqslant b$$

for all $\delta\leqslant\delta_0$, and $g_1(x,y,t)=g_2(x,y,t)=0$. We have for any $|\omega_0(x,y)|\leqslant\delta_0 b$, if $(\omega(x,y,t),\psi(x,y,t))$ is a solution to the equation

$$\begin{aligned}&\omega_t+J(\psi,\omega)+\beta\psi_x=\nu\Delta\omega-r\omega+f(x,y,t,\omega),\\&\Delta\psi=\omega,\end{aligned}$$

with boundary and initial conditions

$$\begin{aligned}&\psi(x,y,t)=0\quad\text{on }\partial D,\\&\omega(x,y,t)=0\quad\text{on }\partial D,\\&\omega(x,y,0)=\omega_0(x,y)\end{aligned}$$

we have

$$\left|\omega(x,y,t)\right|\leqslant\delta_0 e^{(b-r)t}$$

and

$$\left|\psi(x,y,t)\right|\leqslant C\delta_0 e^{(b-r)t},$$

where C depends only on Ω.

Proof. From the remark we see that $\omega(x,y,t)=\psi(x,y,t)=0$ is a periodic solution, from Theorem 2.6, we have the decaying result. □

Finally, the following results show that the equation can have multiple time-periodic solutions.

Theorem 2.9. *Under assumption of Lemma* 2.4*, and $f(x,y,t,0)=g_2(x,y,t,0)=0$. Furthermore, there is $c>0$ such that $f(x,y,t,c)\geqslant rc$, $g_2(x,y,t,c)\geqslant c$. Then the equation*

$$\begin{aligned}&\omega_t+J(\psi,\omega)+\beta\psi_x=\nu\Delta\omega-r\omega+f(x,y,t,\omega),\\&\Delta\psi=\omega,\end{aligned}$$

with boundary and initial conditions

$$\begin{aligned}&\psi(x,y,t)=0\quad\text{on }\partial D,\\&\omega(x,y,t)=g_2(x,y,t,\omega)\quad\text{on }\partial D,\\&\omega(x,y,0)=\omega(x,y,T)\end{aligned}$$

has at least two time-periodic solutions.

Proof. First, the assumption shows that $\omega(x, y, t) = \psi(x, y, t) = 0$ is a periodic solution. Furthermore, the assumption on f indicates that $\underline{\boldsymbol{u}} = (c, \frac{1}{2}y^2c - M)$ is a lower solution for very large M, so we have another periodic solution $M \geqslant \omega(x, y, t) \geqslant c$. $\square$

3. Conclusion

The quasigeostrophic equation studied here models the geophysical fluid dynamics in the case of infinite Rossby deformation radius and with flat bottom and is applicable to planetary-scale solutions. The situation with infinite Rossby deformation radius is equivalent to the rigid-lid approximation. We have shown that the nonlinear operator defined by quasigeostrophic equation is monotonic in the sense that positive solutions exist when the initial and boundary data is positive. We also show that the equation admit multiple periodic solutions. Our result *does not* impose the conditions on β and r, but does require the external forcing is positive or maintain the same sign. On the other hand, the monotonicity of the operator $L(\omega) = J(\psi, \omega) + \beta\psi_x - \nu\Delta\omega + r\omega$, enables one to obtain more properties of the long term dynamical behavior of the solutions, including the stability of the periodic solutions obtained and decaying result for small initial data with homogeneous boundary conditions.

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